

# TORUS COBORDISM OF LENS SPACES

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ABSTRACT. Let  $T^n$  be the real  $n$ -torus group. We show that any 3-dimensional lens space  $L(p; q)$  is  $T^2$ -equivariantly cobordant to zero. We also give some sufficient conditions for higher dimensional lens spaces  $L(p; q_1, \dots, q_n)$  to be  $T^{n+1}$ -equivariantly cobordant to zero using toric topological arguments.

## 1. INTRODUCTION

Cobordism was first introduced by Lev Pontryagin in his geometric works on manifolds, [Pon47]. In early 1950's René Thom [Tho54] showed that cobordism groups could be computed through homotopy theory using the Thom construction, and now we know the oriented, non-oriented and complex cobordism rings completely. On the other hand, even though there have been a lot of developments, the equivariant cobordism rings are not determined for any nontrivial groups. Part of the reason is that the Thom transversality theorem does not hold in equivariant category, and hence the equivariant cobordism can not be reduced to homotopy theory.

In this article we study equivariant cobordism of lens spaces. In particular, Theorem 4.12 gives a sufficient condition for a  $(2n + 1)$ -dimensional lens space  $L(p; q_1, \dots, q_n)$  to be  $T^{n+1}$ -equivariant boundary, where  $T^{n+1}$  is the rank  $n + 1$  real torus group. In particular, Corollary 4.13 shows that if any two integers of  $p, q_1, \dots, q_n$  are relatively prime, then  $L(p; q_1, \dots, q_n)$  is a  $T^{n+1}$ -equivariant boundary, whose nonequivariant version for  $n = 1$  is well-known.

The main tool to get such results is the theory of quasitoric manifolds, which was introduced by Davis and Januskiewicz in [DJ91]. A quasitoric manifold is a closed  $2n$ -dimensional manifold  $M$  with a locally standard  $T^n$  action whose orbit space has the structure of an  $n$ -dimensional simple convex polytope  $P$ . Each codimension one face  $F$  of  $P$ , called a *facet*, corresponds to codimension two submanifolds fixed by a circle subgroup  $S_F^1$  of  $T^n$ , and such information is recorded as a *characteristic function*

$$\lambda: \mathcal{F}(P) \rightarrow \mathbb{Z}^n$$

defined up to sign. Here  $\mathcal{F}(P)$  denotes the set of facets of  $P$ . This characteristic function must satisfy the following *nonsingularity condition*.

**Nonsingularity Condition 1.1.** If the intersection  $F_1 \cap \dots \cap F_\ell$  of  $\ell$  facets of  $P$  is an  $(n - \ell)$ -dimensional facet of  $P$ , then the integral vectors  $\lambda(F_1), \dots, \lambda(F_\ell)$  form a part of basis of  $\mathbb{Z}^n$  for all  $\ell = 1, \dots, n$

Conversely, for a given  $n$ -dimensional simple polytope  $P$  and a function  $\lambda: \mathcal{F}(P) \rightarrow \mathbb{Z}^n$  satisfying the above nonsingularity condition, a quasitoric manifold  $M(P, \lambda)$  with  $P$  as its orbit space and  $\lambda$  as its characteristic function can be constructed. Indeed, the quotient space

$$(1.1) \quad M(P, \lambda) = T^n \times P / \sim,$$

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where  $\sim$  is the equivalence relation defined by

$$(t, p) \sim (s, q) \Leftrightarrow p = q \text{ and } ts^{-1} \in T_F.$$

Here,  $F$  is the unique face of  $P$  containing  $x$  in its relative interior, and if  $F$  is the intersection  $F_1 \cap \cdots \cap F_\ell$  of  $\ell$  facets, then  $T_F$  is the torus subgroup corresponding to the subgroup of  $\mathbb{Z}^n$  generated by the vectors  $\lambda(F_1), \dots, \lambda(F_\ell)$ .

In Section 2, we modify the definition of characteristic function to a *hyper characteristic function*

$$\xi: \mathcal{F}(\Delta^n) \rightarrow \mathbb{Z}^{n+1}$$

defined on the set of facets of the  $n$ -simplex  $\Delta^n$ . Note that the rank of the target group is  $n+1$ , instead of  $n$ . Then, by similar construction to (1.1), we can construct a  $(2n+1)$ -dimensional  $T^{n+1}$ -manifold

$$L(\Delta^n, \xi) = T^{n+1} \times \Delta^n / \sim$$

which is called a *generalized lens space*.

Let  $p > 0$ ,  $q_1, \dots, q_n$  be integers such that  $p$  and  $q_i$  are relatively prime for  $i = 1, \dots, n$ . Let  $f_0, \dots, f_n$  be the facets of  $\Delta^n$ , and consider a hyper characteristic function  $\xi$  defined by

$$\begin{aligned} \xi(f_0) &= (-q_1, \dots, -q_n, p), \text{ and} \\ \xi(f_i) &= e_i, \text{ for } i = 1, \dots, n, \end{aligned}$$

where  $e_i$  is the standard  $i$ th basis vector of  $\mathbb{Z}^{n+1}$ . Then the generalized lens space  $L(\Delta^n, \xi)$  is exactly the usual  $(2n+1)$ -dimensional lens space  $L(p; q_1, \dots, q_n)$ , which gives an alternative construction of a  $(2n+1)$ -dimensional lens space using the technique of toric topology.

Let  $L(p; q_1, \dots, q_n) = L(\Delta^n, \xi)$  be a  $(2n+1)$ -dimensional lens space for a hyper characteristic function  $\xi$  defined as above. Now consider the  $(n+1)$ -simplex  $\Delta^{n+1}$ , and regard  $\Delta^n$  as a facet of it. We would like to extend  $\xi$  to a *rational characteristic function*

$$\eta: (\Delta^{n+1}) \rightarrow \mathbb{Z}^{n+1},$$

i.e.,  $\eta$  satisfy the nonsingularity condition for  $\ell = 1, \dots, n-1$  so that

$$\begin{aligned} \eta(F_0) &= \xi(f_0) = (-q_1, \dots, -q_n, p), \text{ and} \\ \eta(F_i) &= \xi(f_i) = e_i, \text{ for } i = 1, \dots, n, \end{aligned}$$

where  $F_i$  is the facet containing  $f_i$  and not equal to  $\Delta^n$ . Then the space  $M = T^{n+1} \times \Delta^{n+1} / \sim$  constructed similarly to (1.1) is, in general, an orbifold with singularities occurring at the points corresponding to the vertices of  $\Delta^{n+1}$ .

Now consider the *vertex-cut*  $\Delta_V^{n+1}$  of  $\Delta^{n+1}$ , i.e., cutting off a small disjoint  $(n+1)$ -simplex-shaped neighborhoods from each vertex of  $\Delta^{n+1}$ . Then

$$W := \pi^{-1}(\Delta_V^{n+1}) = T^{n+1} \times \Delta_V^{n+1} / \sim \subset M$$

is a  $(2n+2)$ -dimensional manifold with boundary, which consists of  $(2n+1)$  dimensional lens spaces, and in particular one of the boundary components is the lens space  $L(p; q_1, \dots, q_n)$ . Here  $\pi: T^{n+1} \times \Delta^{n+1} / \sim \rightarrow \Delta^{n+1}$  is the map induced from the projection. If the numbers  $p, q_1, \dots, q_n$  satisfy certain number theoretical condition, then we can continue the similar procedure to the other boundary components to show that they are equivariant boundaries, and hence the lens space  $L(p; q_1, \dots, q_n)$  is an equivariant boundary. This is how we get the main result of this article.

## 2. SOME QUOTIENT SPACES OF ODD DIMENSIONAL SPHERES AND LENS SPACES

Let  $p > 0, q_1, \dots, q_n$  be integers such that  $p$  and  $q_i$  are relatively prime for all  $i = 1, \dots, n$ . The  $(2n+1)$ -dimensional lens space  $L(p; q_1, \dots, q_n)$  is the orbit space  $S^{2n+1}/\mathbb{Z}_p$  where  $\mathbb{Z}_p$  action on  $S^{2n+1}$  is defined by

$$\theta: \mathbb{Z}_p \times S^{2n+1} \rightarrow S^{2n+1}$$

$$(k, (z_1, \dots, z_n)) \mapsto (e^{2kq_1\pi\sqrt{-1}/p}z_1, \dots, e^{2kq_n\pi\sqrt{-1}/p}z_n, e^{2k\pi\sqrt{-1}/p}z_{n+1})$$

where  $S^{2n+1} = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \mid |z_1|^2 + \dots + |z_{n+1}|^2 = 1\}$ .

There is an alternative description of a 3-dimensional lens space  $L(p; q)$  as the result of gluing two solid tori via an appropriate homeomorphism of their boundaries, see [OR70]. Moreover it is shown in ([Bro60]) that  $L(p; q)$  is homeomorphic to  $L(p; r)$  if and only if  $r \equiv \pm q \pmod{p}$  or  $qr \equiv \pm 1 \pmod{p}$ . In this section we give another description of  $(2n+1)$ -dimensional lens space  $L(p; q_1, \dots, q_n)$ , and find some sufficient condition for two  $(2n+1)$ -dimensional lens spaces to be diffeomorphic using toric topological techniques.

Let  $\Delta^n = V_0 V_1 \dots V_n$  be an  $n$ -dimensional simplex with vertices  $V_0, \dots, V_n$ . Let  $F_i$  be the facet of  $\Delta^n$  which does not contain  $V_i$ , and let  $\mathcal{F}(\Delta^n)$  denote the set  $\{F_0, \dots, F_n\}$  of facets of  $\Delta^n$ .

**Definition 2.1.** A function  $\xi: \mathcal{F}(\Delta^n) \rightarrow \mathbb{Z}^{n+1}$  is called a *hyper characteristic function* of  $\Delta^n$  if whenever  $F_{i_1} \cap F_{i_2} \cap \dots \cap F_{i_\ell}$  is nonempty, the submodule generated by  $\{\xi(F_{i_1}), \dots, \xi(F_{i_\ell})\}$  is a direct summand of  $\mathbb{Z}^{n+1}$  of rank  $\ell$ . For notational convenience, let us denote  $\xi(F_i)$  by  $\xi_i$  for  $i = 0, \dots, n$ .

**Example 2.2.** Some hyper characteristic functions for triangles are given in Figure 1. It is easy to show that the submodules generated by  $\{(0, 2, 3), (4, 1, 0)\}$ ,  $\{(4, 1, 0), (3, 2, 4)\}$  and  $\{(3, 2, 4), (0, 2, 3)\}$  are direct summands of  $\mathbb{Z}^3$ . We can check that the function given in Figure 1 (b) is a hyper characteristic function if and only if  $p$  is relatively prime to each  $q_1$  and  $q_2$ .

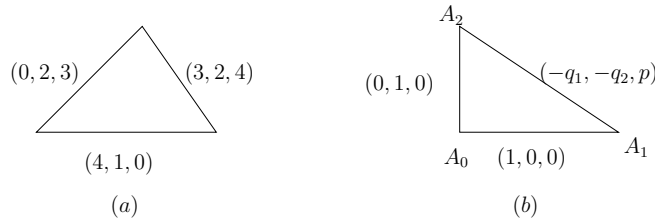


FIGURE 1. Some hyper characteristic functions of triangle.

Of course a hyper characteristic function can be defined for more general simple convex polytope  $P$ , but we only consider the simplex case here.

Let  $F$  be a face of  $\Delta^n$  of codimension  $\ell$ . Then  $F$  is the intersection of a unique collection of  $\ell$  facets  $F_{i_1}, F_{i_2}, \dots, F_{i_\ell}$  of  $\Delta^n$ . Let  $T_F$  be the torus subgroup of  $T^{n+1}$  corresponding to the submodule generated by  $\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_\ell}$  of  $\mathbb{Z}^{n+1}$ . We define an equivalence relation  $\sim$  on the product  $T^{n+1} \times \Delta^n$  as follows.

$$(2.1) \quad (t, x) \sim (u, y) \text{ if and only if } x = y \text{ and } tu^{-1} \in T_F$$

where  $F \subset \Delta^n$  is the unique face containing  $x$  in its relative interior. We denote the quotient space  $(T^{n+1} \times \Delta^n) / \sim$  by  $L(\Delta^n, \xi)$ . Let  $\mathbb{Z}(\Delta^n, \xi) \subset \mathbb{Z}^{n+1}$  be the submodule generated by  $\xi_0, \dots, \xi_n$ , and let  $\mathbb{Z}_\xi(\Delta^n) := \mathbb{Z}^{n+1} / \mathbb{Z}(\Delta^n, \xi)$ . Note that the rank of  $\mathbb{Z}(\Delta^n, \xi) = n$  or  $n+1$ , since  $\xi$  is a hyper characteristic function.

**Proposition 2.3.** *The quotient space  $L(\Delta, \xi)$  is a  $(2n+1)$ -dimensional topological manifold with the natural effective action of  $T^{n+1}$  induced from the group operation on the first factor of  $T^{n+1} \times \Delta^n$ . Furthermore,*

- (1) *if the rank of  $\mathbb{Z}(\Delta^n, \xi)$  is  $n$  then  $L(\Delta^n, \xi)$  is homeomorphic to  $S^1 \times \mathbb{C}P^n$ , and*
- (2) *if the rank of  $\mathbb{Z}(\Delta^n, \xi)$  is  $n+1$  then  $L(\Delta^n, \xi)$  is homeomorphic to  $S^{2n+1}/\mathbb{Z}_\xi(\Delta^n)$ . In particular, if  $\{\xi_0, \dots, \xi_n\}$  form a basis for  $\mathbb{Z}^{n+1}$  then  $L(\Delta^n, \xi)$  is homeomorphic to  $S^{2n+1}$ .*

In the case when rank of  $\mathbb{Z}(\Delta^n, \xi)$  is  $n+1$ , we call the space  $L(\Delta^n, \xi)$  a *generalized lens space* corresponding to the hyper characteristic function  $\xi$  of  $\Delta^n$ . We call  $(\Delta^n, \xi)$  a *combinatorial model* for the generalized lens space.

*Proof.* Let  $U_i = \Delta^n - F_i$  for  $i = 0, 1, \dots, n$ . Then  $U_i$  is diffeomorphic to

$$\mathbb{R}_{\geq 0}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for } i = 1, \dots, n\}.$$

Let  $f^i : U_i \rightarrow \mathbb{R}_{\geq 0}^n$  be a diffeomorphism. Let  $S_{\xi_i}^1$  be the circle subgroup of  $T^{n+1}$  determined by the vector  $\xi_i$  for  $i = 0, \dots, n$ . So  $T^{n+1} = S_i^1 \times S_{\xi_0}^1 \times \dots \times \widehat{S_{\xi_i}^1} \times \dots \times S_{\xi_n}^1$  for some circle subgroup  $S_i^1$  determined by some primitive vector  $v_i \in \mathbb{Z}^{n+1}$ , where  $\widehat{\phantom{x}}$  represents the omission of the circle  $S_{\xi_i}^1$ . Let  $\{e_1, \dots, e_{n+1}\}$  be the standard basis of  $\mathbb{Z}^{n+1}$  over  $\mathbb{Z}$ . Consider the diffeomorphism  $g^i : T^{n+1} \rightarrow T^{n+1}$  defined by  $g^i(v_i) = e_{i+1}$  and  $g^i(\xi_j) = e_{j+1}$  for  $j \neq i$ . So the diffeomorphism  $g^i \times f^i : T^{n+1} \times U_i \rightarrow T^{n+1} \times \mathbb{R}_{\geq 0}^n$  induces an weakly-equivariant homeomorphism from  $(T^{n+1} \times U_i)/\sim$  to  $(T^{n+1} \times \mathbb{R}_{\geq 0}^n)/\sim_e = S_{e_{i+1}}^1 \times ((S_{e_1}^1 \times \dots \times \widehat{S_{e_{i+1}}^1} \times \dots \times S_{e_{n+1}}^1)/\sim_e) \cong S_{e_{i+1}}^1 \times \mathbb{R}^{2n}$  where  $\sim_e$  is the relation  $\sim$  of Lemma 1.6 in [DJ91]. Hence  $L(\Delta, \xi)$  is covered by the  $(2n+1)$ -dimensional open sets  $(T^{n+1} \times U_i)/\sim$ .

Suppose the rank of  $\mathbb{Z}(\Delta^n, \xi)$  is  $n$ . Since  $\xi$  is a hyper characteristic function, the submodule generated by  $\{\xi_0, \dots, \xi_{n-1}\}$  is a direct summand of rank  $n$ . Then the vector  $\xi_n$  is contained in the subgroup  $\langle \xi_0, \dots, \xi_{n-1} \rangle$  and there exists  $v \in \mathbb{Z}^{n+1}$  such that  $\{\xi_0, \dots, \xi_{n-1}, v\}$  is a basis of  $\mathbb{Z}^{n+1}$ . So by considering some automorphism on  $\mathbb{Z}^{n+1}$ , if necessary, we may regard  $\xi$  as a characteristic function of  $\Delta^n$  in the sense of [DJ91]. Let  $S_v^1$  be the circle subgroup of  $T^{n+1}$  determined by  $v$ . So  $(T^{n+1} \times \Delta^n)/\sim \cong S_v^1 \times (T_{V_n} \times \Delta^n)/\sim \cong S_v^1 \times \mathbb{C}P^n$ .

Assume the rank of  $\mathbb{Z}(\Delta^n, \xi)$  is  $n+1$ . Let  $\{e_1, \dots, e_{n+1}\}$  be the standard basis of  $\mathbb{Z}^{n+1}$  over  $\mathbb{Z}$ . Define  $\xi^s(F_i) = e_{i+1} = \xi_i^s$  called the standard hyper characteristic function of  $\Delta^n$ . Consider the standard action of  $T^{n+1}$  on  $\mathbb{C}^{n+1}$ . The orbit map  $\pi_s : \mathbb{C}^{n+1} \rightarrow \mathbb{R}_{\geq 0}^{n+1}$  of this action is given by  $(z_1, \dots, z_{n+1}) \rightarrow (|z_1|, \dots, |z_{n+1}|)$ . Let  $H = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}_{\geq 0}^{n+1} : x_1 + \dots + x_{n+1} = 1\}$ . Then  $H$  is diffeomorphic as manifold with corners to  $\Delta^n$ . Facets of  $H$  are  $H_i = \{(x_1, \dots, x_{n+1}) \in H : x_i = 0\}$  for  $i = 1, \dots, n+1$ . The isotropy subgroup of  $\pi_s^{-1}(H_i)$  is the  $i$ th circle subgroup of  $T^{n+1}$ . So we get a hyper characteristic function on  $H$  which is nothing but the standard one. Hence it is clear that  $S^{2n+1} = \pi_s^{-1}(H) \cong (T^{n+1} \times H)/\sim_s \cong (T^{n+1} \times \Delta^n)/\sim_s = L(\Delta^n, \xi^s)$ , where  $\sim_s$  is the equivalence relation  $\sim$  defined in (2.1) corresponding to the standard hyper characteristic function  $\xi^s$ . Consider the map  $\beta : \mathbb{Z}^{n+1} \rightarrow \mathbb{Z}^{n+1}$  defined by  $\beta(e_i) = \xi_{i-1}^s = \xi_{i-1}$  for  $i = 1, \dots, n+1$ . Let  $\mathbb{Z}_{\mathbb{R}}^{n+1} = \mathbb{Z}^{n+1} \otimes_{\mathbb{Z}} \mathbb{R}$ . Since the rank of  $\mathbb{Z}(\Delta^n, \xi)$  is  $n+1$ , the map  $\beta$  induces a surjective homomorphism

$$\overline{\beta} : T^{n+1} \cong \mathbb{Z}_{\mathbb{R}}^{n+1} / \text{Im}(\beta) \rightarrow \mathbb{Z}_{\mathbb{R}}^{n+1} / \mathbb{Z}^{n+1} \cong T^{n+1}$$

defined by  $v + \text{Im}(\beta) \rightarrow v + \mathbb{Z}^{n+1}$ . The kernel of  $\overline{\beta}$  is  $\mathbb{Z}^{n+1} / \text{Im}(\beta) = \mathbb{Z}_\xi(\Delta^n)$ , a finite subgroup of  $T^{n+1}$ . From the definition of  $\sim_s$  and  $\sim$  it is clear that  $\overline{\beta} \times \text{id} : T^{n+1} \times \Delta^n \rightarrow T^{n+1} \times \Delta^n$  induces a surjective map

$$f_\beta : S^{2n+1} \rightarrow L(\Delta^n, \xi)$$

defined by  $f_\beta([t, x]^{\sim s}) = [\bar{\beta}(t), x]^{\sim}$  on the equivalence classes. The finite group  $\mathbb{Z}_\xi(\Delta^n)$  has a natural free and smooth action on  $T^{n+1}$  induced by the group operation. This induces a smooth action of  $\mathbb{Z}_\xi(\Delta^n)$  on  $S^{2n+1}$ . Since  $\bar{\beta}$  is a covering homomorphism with the finite covering group  $\mathbb{Z}_\xi(\Delta^n)$ ,  $L(\Delta^n, \xi)$  is homeomorphic to the quotient space  $S^{2n+1}/\mathbb{Z}_\xi(\Delta^n)$ .  $\square$

**Remark 2.4.** If  $\{\xi_0, \dots, \xi_n\}$  is a basis of  $\mathbb{Z}^{n+1}$  over  $\mathbb{Z}$  then the manifold  $L(\Delta^n, \xi)$  is the sphere  $S^{2n+1}$ , the moment angle manifold of  $\Delta^n$ , where the natural action of  $T^{n+1}$  may differ from the standard action on  $S^{2n+1}$  by an automorphism of  $T^{n+1}$ .

We now consider generalized lens spaces with a particular type of hyper characteristic functions. Let  $p > 0, q_1, \dots, q_n$  be integers such that  $p$  is relatively prime to each  $q_i$  for  $i = 1, \dots, n$ . Define a function  $\xi : \mathcal{F}(\Delta^n) \rightarrow \mathbb{Z}^{n+1}$  by  $\xi(F_i) = e_i$  for  $i = 1, \dots, n$  and  $\xi(F_0) = (-q_1, -q_2, \dots, -q_n, p)$ . So  $\xi$  is a hyper characteristic function of  $\Delta^n$ . The rank of the submodule generated by  $\{\xi(F_i) \mid i = 0, \dots, n\}$  is  $n+1$ . In this case the surjective homomorphism  $\bar{\beta} : T^{n+1} \rightarrow T^{n+1}$  induced by  $\xi$  is given by,

$$(2.2) \quad (t_1, \dots, t_{n+1}) \rightarrow (t_1 t_{n+1}^{-q_1}, \dots, t_n t_{n+1}^{-q_n}, t_{n+1}^p).$$

Then  $\mathbb{Z}_\xi(\Delta^n) \cong \{(t, \dots, t_{n+1}) \in T^{n+1} \mid t_{n+1}^{-q_i} t_i = 1 \text{ for } i = 1, \dots, n \text{ and } t_{n+1}^p = 1\}$ . Let  $\omega$  be the  $p$ -th root of unity. So  $\mathbb{Z}_\xi(\Delta^n) \cong \{(\omega^{q_1}, \dots, \omega^{q_n}, \omega) \in T^{n+1}\} \cong \mathbb{Z}_p$ . The  $\mathbb{Z}_\xi(\Delta^n)$ -action on  $S^{2n+1}$  induced by the group operation on  $T^{n+1}$  is nothing but the following:

$$(\omega^{q_1}, \dots, \omega^{q_n}, \omega) \times (z_1, z_2, \dots, z_{n+1}) \rightarrow (\omega^{q_1} z_1, \dots, \omega^{q_n} z_n, \omega z_{n+1})$$

where  $(z_1, \dots, z_{n+1}) \in S^{2n+1}$ . Hence  $L(\Delta^n, \xi)$  is a usual  $(2n+1)$ -dimensional lens space  $L(p; q_1, \dots, q_n)$ .

We now discuss some classification results of  $(2n+1)$ -dimensional lens spaces. For an automorphism  $\delta$  on  $T^{n+1}$  and a combinatorial model  $(\Delta^n, \xi)$  of a generalized lens space  $L$ , the  $\delta$ -translation of  $(\Delta^n, \xi)$  is the combinatorial model  $(\Delta^n, \delta(\xi))$  where  $\delta(\xi) : \mathcal{F}(\Delta^n) \rightarrow \mathbb{Z}^{n+1}$  is the hyper characteristic function such that  $\delta(\xi)(F_i)$  is the vector in  $\mathbb{Z}^{n+1}$  up to sign determined by the circle subgroup  $\delta(T_{\xi_i}^1)$ . Here  $T_{\xi_i}^1$  is the circle subgroup of  $T^{n+1}$  determined by the vector  $\xi_i$ .

A diffeomorphism  $g : L_1 \rightarrow L_2$  between two  $T^{n+1}$ -manifolds  $L_1$  and  $L_2$  is  $\delta$ -equivariant (or *weakly-equivariant*) for an automorphism  $\delta$  of  $T^{n+1}$  if  $g(t \cdot x) = \delta(t) \cdot g(x)$  for all  $(t, x) \in T^{n+1} \times L_1$ . The arguments of the proof of the following lemma is similar to classification of quasitoric manifolds, see the proof of Lemma 1.8 of [DJ91].

**Lemma 2.5.** *Let  $L_1$  and  $L_2$  be two lens spaces with combinatorial models  $(\Delta^n, \xi^1)$  and  $(\Delta^n, \xi^2)$  respectively. Then  $L_1$  and  $L_2$  are  $\delta$ -equivariantly diffeomorphic if and only if  $(\Delta^n, \xi^2)$  is a  $\delta$ -translation of  $(\Delta^n, \xi^1)$ .*

The following lemma gives a classification of lens spaces  $L(p; q_1, \dots, q_n)$  upto diffeomorphisms. From the integers  $q_1, \dots, q_n$  we obtain the integers  $r_1, \dots, r_n$  as follows. Let  $\mathbf{q} := (-q_1, \dots, -q_n)^t$ . Choose any  $B \in SL(n, \mathbb{Z})$ , and let  $\mathbf{a} = (a_1, \dots, a_n)^t := B\mathbf{q}$ . Then consider a vector  $\mathbf{a}' = (a'_1, \dots, a'_n)^t$  such that  $\mathbf{a}' \equiv \mathbf{a} \pmod{p}$ . Now let  $\mathbf{q}' := (q'_1, \dots, q'_n)^t = -B^{-1}\mathbf{a}'$ , and choose  $\mathbf{r} := (r_1, \dots, r_n)^t$  to be  $\mathbf{r} \equiv \mathbf{q}' \pmod{p}$ .

**Lemma 2.6.** *Let  $p (> 0), q_1, \dots, q_n$  be integers such that  $p$  is relatively prime to each  $q_i$ . Let  $r_i$  for  $i = 1, \dots, n$  be the integers obtained as above. Then two lens spaces  $L(p; q_1, \dots, q_n)$  and  $L(p; r_1, \dots, r_n)$  are diffeomorphic.*

*Proof.* Let  $\xi$  be a hyper characteristic function of  $\Delta^n$  defined by  $\xi_0 = (-q_1, \dots, -q_n, p)$  and  $\xi_i = e_i$  for  $i = 1, \dots, n$ . So  $L(\Delta^n, \xi) = L(p; q_1, \dots, q_n)$ . Let  $B = (b_{ij}) \in SL(n, \mathbb{Z})$  and  $\mathbf{a} = (a_1, \dots, a_n)^t = B\mathbf{q}$ . Consider the map  $\delta : \mathbb{Z}^{n+1} \rightarrow \mathbb{Z}^{n+1}$  represented by the matrix

$$\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}.$$

Then  $\delta$  induces an automorphism of  $T^{n+1}$ , which is denoted by the same  $\delta$ . Consider the hyper characteristic function  $\delta(\xi)$ , defined by  $\delta(\xi)_i = \delta(\xi_i)$  for  $i = 0, \dots, n$ . Then  $\delta(\xi)$  induces a surjective homomorphism  $T^{n+1} \rightarrow T^{n+1}$ , defined by

$$(t_1, \dots, t_n, t_{n+1}) \rightarrow (t_1^{b_{11}} \dots t_n^{b_{1n}} t_{n+1}^{a_1}, \dots, t_1^{b_{n1}} \dots t_n^{b_{nn}} t_{n+1}^{a_n}, t_{n+1}^p).$$

Let  $\mathbf{a}' = (a'_1, \dots, a'_n)^t \equiv \mathbf{a} \pmod{p}$ . The kernel of this map is given by

$$\mathbb{Z}_{\delta(\xi)}(\Delta^n) \cong \{(t_1, \dots, t_n, t_{n+1}) \in T^{n+1} \mid t_1^{b_{i1}} \dots t_n^{b_{in}} t_{n+1}^{a'_i} = 1 \text{ for } i = 1, \dots, n \text{ and } t_{n+1}^p = 1\}.$$

Considering the Lie algebra of  $\mathbb{Z}_{\delta(\xi)}(\Delta^n)$ , we need to find  $(x_1, \dots, x_{n+1}) \in \mathbb{Z}$  such that

$$(2.3) \quad b_{i1}x_1 + \dots + b_{in}x_n + a'_i x_{n+1} = 0 \text{ and } px_{n+1} = 0 \text{ for } i = 1, \dots, n.$$

Namely,

$$B \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = - \begin{pmatrix} a'_1 \\ \vdots \\ a'_n \end{pmatrix} x_{n+1}.$$

Then

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = -B^{-1} \begin{pmatrix} a'_1 \\ \vdots \\ a'_n \end{pmatrix} x_{n+1} = \begin{pmatrix} q'_1 \\ \vdots \\ q'_n \end{pmatrix} x_{n+1}.$$

So  $x_i = q'_i x_{n+1}$ , i.e.,  $t_i = t_{n+1}^{q'_i}$ . Let  $r_i \equiv q'_i \pmod{p}$ . Hence  $\mathbb{Z}_{\delta(\xi)}(\Delta^n) \cong \{(\alpha^{r_1}, \dots, \alpha^{r_n}, \alpha) \in T^{n+1} : \alpha^p = 1\}$ .

The automorphism  $\delta$  induces a  $\delta$ -equivariant diffeomorphisms  $\bar{\delta} : S^{2n+1} \cong L(\Delta^n, \xi^s) \rightarrow L(\Delta^n, \delta(\xi^s)) \cong S^{2n+1}$  and  $\overline{\delta(\xi)} : L(\Delta^n, \xi) \rightarrow L(\Delta^n, \delta(\xi))$ , by Lemma 2.5. Here  $\xi^s$  is the standard hyper characteristic function defined in the proof of Proposition 2.3. Clearly, the following diagram is commutative where verticle arrows are orbit maps of the actions of  $\mathbb{Z}_\xi(\Delta^n)$  and  $\mathbb{Z}_{\delta(\xi)}(\Delta^n)$  on  $S^{2n+1}$ .

$$(2.4) \quad \begin{array}{ccc} S^{2n+1} & \xrightarrow{\bar{\delta}} & S^{2n+1} \\ \downarrow & & \downarrow \\ L(\Delta^n, \xi) & \xrightarrow{\overline{\delta(\xi)}} & L(\Delta^n, \delta(\xi)). \end{array}$$

Since  $\mathbb{Z}_\xi(\Delta^n)$  acts freely on  $S^{2n+1}$ ,  $\mathbb{Z}_{\delta(\xi)}(\Delta^n)$  acts freely on  $S^{2n+1}$ . So  $L(\Delta^n, \delta(\xi)) \cong L(p; r_1, \dots, r_n)$ . Hence  $L(p; q_1, \dots, q_n)$  is diffeomorphic to  $L(p; r_1, \dots, r_n)$ .  $\square$

**Example 2.7.** Consider the hyper characteristic functions of a triangle  $\Delta^2$  given in Figure 2. The hyper characteristic function  $\xi^2$  in (b) is the  $\delta$ -translation of the hyper characteristic function  $\xi^1$  in (a), where  $\delta$  is represented by

$$\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \text{ where } B = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}.$$

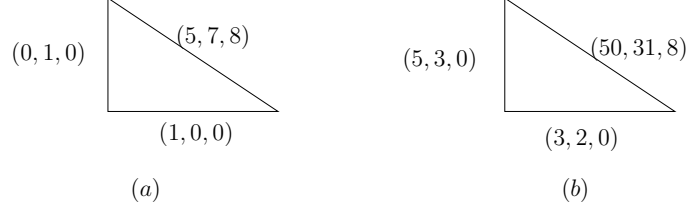
So

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 50 \\ 31 \end{pmatrix} \equiv \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} = \mathbf{a}' \pmod{8}$$

Then

$$\mathbf{q}' = \begin{pmatrix} q'_1 \\ q'_2 \end{pmatrix} = -B^{-1} \mathbf{a}' = - \begin{pmatrix} 3 & -5 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ -5 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \pmod{8}.$$

So  $\mathbb{Z}_{\xi^1}(\Delta^2) = \{(t^{-5}, t^{-7}, t) \in T^3 \mid t^8 = 1\}$  and  $\mathbb{Z}_{\xi^2}(\Delta^2) = \{(t^1, t^3, t) \in T^3 \mid t^8 = 1\}$ . Hence by Lemma 2.6, the lens spaces  $L(\Delta^2, \xi^1) = L(8; -5, -7)$  and  $L(\Delta^2, \xi^2) = L(8; 1, 3)$  are diffeomorphic.

FIGURE 2. Hyper characteristic functions  $\xi^1$  and  $\xi^2$  of a triangle  $\triangle^2$ .

## 3. MANIFOLDS WITH LENS SPACES BOUNDARY

Let  $Q$  be an  $n$ -dimensional simple convex polytope in  $\mathbb{R}^n$  with facets  $F_1, \dots, F_m$  and vertices  $V_1, \dots, V_k$ . Let  $\mathcal{F}(Q)$  denote the set  $\{F_1, \dots, F_m\}$  of the facets of  $Q$ .

**Definition 3.1.** A function  $\eta : \mathcal{F}(Q) \rightarrow \mathbb{Z}^n$  is called a *rational characteristic function* if the set of vectors  $\{\eta(F_{i_1}), \dots, \eta(F_{i_\ell})\}$  form a part of a basis of  $\mathbb{Z}^n$  whenever the intersection of the facets  $\{F_{i_1}, \dots, F_{i_\ell}\}$  is an  $(n - \ell)$ -dimensional face of  $Q$ , where  $n - \ell > 0$ . The vectors  $\eta_i := \eta(F_i)$  for  $i = 1, \dots, m$  are called *rational characteristic vectors*.

Note that the definition of a rational characteristic function is same as that of a characteristic function of a quasitoric manifold in [DJ91] except when  $\ell = 0$ , i.e., when  $F_{i_1} \cap \dots \cap F_{i_n}$  is a vertex of the polytope  $Q$ .

**Example 3.2.** Clearly the function given in the Figure 3 (a) is a rational characteristic function of a rectangle. We can check that the function given in Figure 3 (b) is a rational characteristic function of the tetrahedron if and only if any two integers of  $\{q_1, q_2, p\}$  are relatively prime.

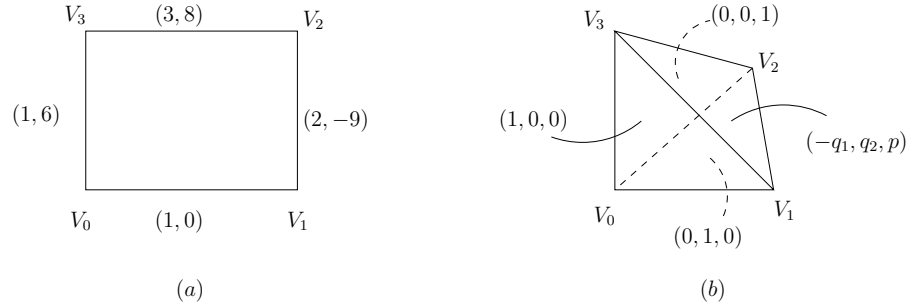


FIGURE 3. Some rational characteristic functions of rectangle and tetrahedron.

Let  $\eta : \mathcal{F}(Q) \rightarrow \mathbb{Z}^n$  be a rational characteristic function of  $Q$ . Let  $F$  be a face of  $Q$  of codimension  $\ell$  with  $0 < \ell < n$ . Since  $Q$  is a simple polytope,  $F$  is the intersection of a unique collection of  $\ell$  many facets  $F_{i_1}, F_{i_2}, \dots, F_{i_\ell}$  of  $Q$ . Let  $T_F$  be the torus subgroup of  $T^n$  corresponding to the submodule generated by  $\eta_{i_1}, \eta_{i_2}, \dots, \eta_{i_\ell}$  in  $\mathbb{Z}^n$ . We assume  $T_Q = 1$  and  $T_{V_i} = T^n$  for each vertex  $V_i$  of  $Q$ . We define an equivalence relation  $\sim_b$  on the product  $T^n \times Q$  as follows:

$$(3.1) \quad (t, x) \sim_b (u, y) \text{ if and only if } x = y \text{ and } tu^{-1} \in T_F$$

where  $F \subset Q$  is the unique face containing  $x$  in its relative interior. We denote the quotient space  $(T^n \times Q) / \sim_b$  by  $X(Q, \eta)$  and the equivalence class of  $(t, x)$  by  $[t, x]^{\sim_b}$ . The space  $X(Q, \eta)$  is a manifold if and only if the set of vectors  $\{\eta(F_{i_1}), \dots, \eta(F_{i_n})\}$  form a basis of  $\mathbb{Z}^n$  whenever  $F_{i_1} \cap \dots \cap F_{i_n}$  is a vertex of  $Q$ . In this case, the space  $X(Q, \eta)$  is called a *quasitoric manifold* which was introduced by M. Davis and T. Januszkiewicz in

[DJ91]. The space  $X(Q, \eta)$  is a quasitoric orbifold if the rank of the submodule generated by  $\{\eta(F_{i_1}), \dots, \eta(F_{i_n})\}$  is  $n$  whenever  $F_{i_1} \cap \dots \cap F_{i_n}$  is a vertex of  $Q$ , see the Section 2 in [PS10]. Let  $\pi : X(Q, \eta) \rightarrow Q$  be the projection map defined by  $\pi([t, x]^{\sim_b}) = x$ .

We can construct  $T^n$ -manifold with boundary from the orbifold  $X(Q, \eta)$  as follows. Cut off a neighborhood of each vertex  $V_i, i = 1, 2, \dots, k$  of the simple polytope  $Q$  by an affine hyperplane  $H_i, i = 1, 2, \dots, k$  in  $\mathbb{R}^n$  such that  $H_i \cap H_j \cap Q$  are empty sets for  $i \neq j$ . We call this operation the *vertex cut* of  $Q$ . Then the remaining subset of the convex polytope  $Q$  is an  $n$ -dimensional simple convex polytope, denoted by  $Q_V$ . Then the facet  $\Delta_{V_i}^{n-1} := Q \cap H_i (= Q_V \cap H_i)$  of  $Q_V$  is an  $(n-1)$ -dimensional simplex for each  $i = 1, 2, \dots, k$ . We restrict the equivalence relation  $\sim_b$  in (3.1) to  $T^n \times Q_V$ , and consider the quotient space

$$(3.2) \quad W(Q_V, \eta) = (T^n \times Q_V) / \sim_b \subset X(Q, \eta).$$

The natural action of  $T^n$  on  $W(Q_V, \eta)$  is induced by the group operation in  $T^n$ .

**Lemma 3.3.** *Let  $\eta$  be a rational characteristic function of a simple polytope  $Q$ , and let  $Q_V$  be the vertex cut of  $Q$ . Then  $W(Q_V, \eta)$  in (3.2) is an oriented  $2n$ -dimensional  $T^n$ -manifold with boundary, whose boundary is a disjoint union of  $(2n-1)$ -dimensional generalized lens spaces.*

*Proof.* For any  $[t, x]^{\sim_b} \in W(Q_V, \eta)$  we show that  $[t, x]^{\sim_b}$  has a neighborhood which is homeomorphic to an open subset of  $\mathbb{R}^{2n-1} \times \mathbb{R}_{\geq 0}$ . If  $x$  belongs to the interior of  $Q_V$ , there is a neighborhood  $U_x$  of  $x$  in  $Q_V$  which is homeomorphic to an  $n$ -dimensional open ball in  $\mathbb{R}^n$ . Clearly,  $\pi^{-1}(U_x) = T^n \times U_x / \sim_b = T^n \times U_x$ . So the claim is true in this case.

Let  $x$  belong to the relative interior of a codimension  $\ell > 0$  face  $F$  of  $Q_V$  where  $F$  is not a face of  $\Delta_{V_j}^{n-1}$  for  $j = 1, \dots, k$ . So there is a neighborhood  $U_x$  in  $Q_V$  of  $x$  which is diffeomorphic to  $\mathbb{R}^{n-\ell} \times \mathbb{R}_{\geq 0}^\ell$ . The facets of  $\mathbb{R}^{n-\ell} \times \mathbb{R}_{\geq 0}^\ell$  are  $\{\mathbb{R}^{n-\ell} \times H_j \mid j = 1, \dots, \ell\}$  where  $H_j$  is the facet of  $\mathbb{R}_{\geq 0}^\ell$  with  $j$ -th coordinate zero. Let  $F = F_{i_1} \cap \dots \cap F_{i_\ell}$  be the unique intersection of  $\ell$  many facets of  $Q_V$ . By Definition 3.1, the submodule  $\mathbb{Z}(F)$  generated by  $\{\eta(F_{i_1}), \dots, \eta(F_{i_\ell})\}$  is a direct summand of  $\mathbb{Z}^n$  of rank  $\ell$ . So  $\{\eta_{i_1}, \dots, \eta_{i_\ell}\}$  is a basis of  $\mathbb{Z}(F)$  and  $\mathbb{Z}^n \cong \mathbb{Z}^{n-\ell} \oplus \mathbb{Z}(F)$ . This direct sum decomposition gives  $T^n \cong T^{n-\ell} \times T_F$ . Suppose the diffeomorphism  $\phi : U_x \rightarrow \mathbb{R}^{n-\ell} \times \mathbb{R}_{\geq 0}^\ell$  sends  $F_{i_j} \cap U_x$  to  $\mathbb{R}^{n-\ell} \times H_j$  for all  $j = 1, 2, \dots, \ell$ . Define a rational characteristic function  $\eta_x$  on the set of all facets of  $\mathbb{R}^{n-\ell} \times \mathbb{R}_{\geq 0}^\ell$  by  $\eta_x(\mathbb{R}^{n-\ell} \times H_j) = \eta_{i_j}$  for all  $j = 1, 2, \dots, \ell$ . We define an equivalence relation  $\sim_x$  on  $T^n \times \mathbb{R}^{n-\ell} \times \mathbb{R}_{\geq 0}^\ell$  as follows.

$$(3.3) \quad (t, y_1, z_1) \sim_x (u, y_2, z_2) \text{ if and only if } (y_1, z_1) = (y_2, z_2) \text{ and } tu^{-1} \in T_{F'}.$$

where  $F'$  is the unique face of  $\mathbb{R}^{n-\ell} \times \mathbb{R}_{\geq 0}^\ell$  containing  $(y_1, z_1)$  in its relative interior. So for each  $y \in \mathbb{R}^{n-\ell}$  the restriction of  $\eta_x$  on  $\{\{y\} \times H_j \mid j = 1, 2, \dots, \ell\}$  defines a characteristic function on the set of facets of  $\{y\} \times \mathbb{R}_{\geq 0}^\ell$ , see [DJ91]. From the constructive definition of quasitoric manifold given in [DJ91] it is clear that the quotient space  $\{y\} \times (T_F \times \mathbb{R}_{\geq 0}^\ell) / \sim_x$  is homeomorphic to  $\{y\} \times \mathbb{R}^{2\ell}$ . Hence the quotient space

$$(T^n \times \mathbb{R}^{n-\ell} \times \mathbb{R}_{\geq 0}^\ell) / \sim_x = T^{n-\ell} \times \mathbb{R}^{n-\ell} \times (T_F \times \mathbb{R}_{\geq 0}^\ell) / \sim_x \cong T^{n-\ell} \times \mathbb{R}^{n-\ell} \times \mathbb{R}^{2\ell}.$$

Since the maps  $\pi : (T^n \times U_x) \rightarrow (T^n \times U_x) / \sim_b$  and  $\pi_x : (T^n \times \mathbb{R}^{n-\ell} \times \mathbb{R}_{\geq 0}^\ell) \rightarrow (T^n \times \mathbb{R}^{n-\ell} \times \mathbb{R}_{\geq 0}^\ell) / \sim_x$  are quotient maps and  $\phi$  is a diffeomorphism, the following commutative diagram ensure that the lower horizontal map  $\phi_x$  is a homeomorphism.

$$(3.4) \quad \begin{array}{ccc} (T^n \times U_x) & \xrightarrow{id \times \phi} & (T^n \times \mathbb{R}^{n-\ell} \times \mathbb{R}_{\geq 0}^\ell) \\ \pi \downarrow & & \downarrow \pi_x \\ (T^n \times U_x) / \sim_b & \xrightarrow{\phi_x} & (T^n \times \mathbb{R}^{n-\ell} \times \mathbb{R}_{\geq 0}^\ell) / \sim_x \xrightarrow{\cong} T^{n-\ell} \times \mathbb{R}^{n-\ell} \times \mathbb{R}^{2\ell} \end{array}$$

Let  $x$  belongs to the relative interior of a face  $F$  of  $Q_V$  where  $F \subset \Delta_{V_j}^{n-1}$  for some  $j \in \{1, \dots, k\}$ . Since  $Q_V$  is a simple polytope in  $\mathbb{R}^n$ , there exists a collar neighborhood  $U_x$  of  $\Delta_{V_j}^{n-1}$  in  $Q_V$ . So there is a diffeomorphism  $f_j : U_x \rightarrow \Delta_{V_j}^{n-1} \times [0, 1]$ . Let  $V_j = F_{j_1} \cap \dots \cap F_{j_n}$ . So the facets of  $\Delta_{V_j}^{n-1}$  are  $\Delta_{V_j}^{n-1} \cap F_{j_1}, \dots, \Delta_{V_j}^{n-1} \cap F_{j_n}$ . The restriction of the rational characteristic function  $\eta$  on the facets of  $\Delta_{V_j}^{n-1}$  is given by  $\xi_j(\Delta_{V_j}^{n-1} \cap F_{j_\ell}) = \eta_{j_\ell}$  for  $\ell = 1, \dots, n$ . By definition of  $\eta$ , we can see that  $\xi_j$  is a hyper characteristic function on  $\Delta_{V_j}^{n-1}$ . So from Proposition 2.3 the quotient  $(T^n \times \Delta_{V_j}^{n-1})/\sim$  is a  $(2n-1)$ -dimensional generalized lens space, where  $\sim$  is the equivalence relation defined in (2.1). From the equivalence relation  $\sim_b$  in (3.1), it is clear that  $f_j$  induces a homeomorphism  $h_j : (T^n \times U_x)/\sim_b \rightarrow ((T^n \times \Delta_{V_j}^{n-1})/\sim) \times [0, 1]$ . So the claim is true for this case also.

From the above discussion, we get  $W(Q_V, \eta) = \bigcup_{x \in Q_V} (T^n \times U_x)/\sim$ . Hence  $W(Q_V, \eta)$  is a manifold with boundary where the boundary is the disjoint union of generalized lens spaces  $\{L(\Delta_{V_j}^{n-1}, \xi_j) : j = 1, \dots, k\}$ . Clearly orientations of  $T^n$  and  $Q$  induce an orientation of  $W(Q_V, \eta)$ . We consider the standard orientation of  $T^n$  and the orientation of  $Q$  induced from the ambient space  $\mathbb{R}^n$ .  $\square$

**Remark 3.4.** If a vertex  $V_j = F_{j_1} \cap \dots \cap F_{j_n}$  of  $Q$  such that the set of vectors  $\{\eta(F_{j_1}), \dots, \eta(F_{j_n})\}$  form a basis of  $\mathbb{Z}^n$ , then  $(T^n \times \Delta_{V_j}^{n-1})/\sim$  is  $\delta$ -equivariantly homeomorphic to  $S^{2n-1}$  with the standard  $T^n$  action for some automorphism  $\delta$  of  $T^n$ .

#### 4. TORUS COBORDISM OF $L(p; q_1, \dots, q_n)$

In this section we discuss the  $T^{n+1}$ -equivariant cobordism of lens spaces  $L(p; q_1, \dots, q_n)$ . We recall the definition of  $T^k$ -equivariant cobordism for  $T^k$ -manifolds where  $k$  is a positive integer.

**Definition 4.1.** Two same dimensional oriented closed smooth manifolds  $M_1$  and  $M_2$  with effective  $T^k$ -actions are said to be  $T^k$ -equivariantly cobordant if there exists an oriented  $T^k$  manifold  $W$  with boundary  $\partial W$  such that  $\partial W$  is equivariantly diffeomorphic to  $M_1 \sqcup (-M_2)$  under an orientation preserving diffeomorphism. Here  $-M_2$  represents the reverse orientation of  $M_2$ . When a  $T^k$ -manifold  $M$  is the boundary of an oriented  $T^k$ -manifold with boundary,  $M$  is called  $T^k$ -equivariantly null cobordant.

From the definition of the manifold  $L(\Delta^n, \xi)$  it is clear that  $T^{n+1}$ -action depends on the characteristic function  $\xi$ . We denote the equivariant cobordism class of  $L(p; q_1, \dots, q_n)$  by  $[L(p; q_1, \dots, q_n)]_\delta$  where  $\delta$  represents the action. We discuss the equivariant cobordism of 3-dimensional, 5-dimensional, and higher dimensional lens spaces seperately in the following subsections. Given  $a_0, \dots, a_k \in \mathbb{Z}$ , we denote greatest common divisor of them by  $\gcd\{a_0, \dots, a_k\}$ .

**4.1. Cobordism of  $L(p, q)$ , that is when  $n = 1$ .** In this case  $\Delta^1$  is an 1-dimensional simplex, that is the closed interval  $[0, 1]$ . Define  $\xi : \{\{0\}, \{1\}\} \rightarrow \mathbb{Z}^2$  by  $\xi(\{0\}) = (1, 0)$  and  $\xi(\{1\}) = (-q, p)$  where  $\gcd\{q, p\} = 1$  with  $0 \leq |q| < p$ . From the Section 2 of [OR70] we get that the space  $L(\Delta^1, \xi)$  is the lens space  $L(p, q)$  with the natural  $T^2$  action coming from the group operation on the first factor of  $T^2 \times \Delta^1$ .

**Lemma 4.2.** Let  $(a, b), (c, d) \in \mathbb{Z}^2$  such that  $|\det\{(a, b), (c, d)\}| = r > 1$  and  $\gcd\{a, b\} = 1 = \gcd\{c, d\}$ . Then there exists  $(e, f) \in \mathbb{Z}^2$  such that  $|\det\{(a, b), (e, f)\}| = 1$ ,  $\gcd\{e, f\} = 1$  and  $|\det\{(c, d), (e, f)\}| < r$ .

*Proof.* First we prove when  $(a, b) = (1, 0)$ . Then  $|\det[(1, 0), (c, d)]| = r = |d|$ . Since  $c, d$  are relatively prime and  $r > 1$ ,  $d \neq 0, \pm 1$  and either  $c > d$  or  $d > c$ . Let  $c > 0, d > 0$  and  $c > d$ . Then  $r$  is the area of the parallelogram  $P$  in  $\mathbb{R}^2$  with vertices  $V_1 = (0, 0)$ ,  $V_2 = (1, 0)$ ,

$V_3 = (c+1, d)$  and  $V_4 = (c, d)$ . Clearly the length of  $\{y = 1\} \cap P$  is 1. So  $\{y = 1\} \cap P \cap \mathbb{Z}^2$  is nonempty. It may contains only one point  $(u, 1)$ , since  $c, d$  are relatively primes and the intersection  $\{y = 1\} \cap \{cy = dx\} \cap \mathbb{Z}^2$  is empty. From the elementary geometry we get that the area  $|\det\{(c, d), (u, 1)\}| = |c - du|$  of the parallelogram  $P_1$  with vertices  $(0, 0)$ ,  $(u, 1)$ ,  $(c+u, d+1)$  and  $(c, d)$  is less than  $r$ . Also  $|\det\{(1, 0), (u, 1)\}| = 1$  and  $u, 1$  are relatively primes. For other possible value of  $c, d$ , we can prove similarly.

We now prove the case when  $a, b$  are arbitrary relatively prime integers. Since  $a, b$  are relatively prime, there exists  $(x, y) \in \mathbb{Z}^2$  such that  $\det\{(a, b), (x, y)\} = 1$ . So there exists  $A \in SL(2, \mathbb{Z})$  with  $A(a, b) = (1, 0)$  and  $A(x, y) = (0, 1)$ . Note that if  $\gcd\{c, d\} = 1$  then  $\gcd\{(a_{11}c + a_{12}d), (a_{21}c + a_{22}d)\} = 1$  for any  $(a_{ij}) \in SL(2, \mathbb{Z})$ . Using this and the previous arguments we can prove the lemma for general case.  $\square$

**Lemma 4.3.** *Let  $p, q$  be two relatively prime integers and  $0 < q < p$ . Then there exists a sequence of pairs  $(q_1, p_1), \dots, (q_k, p_k)$  where  $|\det\{(q_i, p_i), (q_{i+1}, p_{i+1})\}| = 1$  for all  $i = 1, \dots, k-1$  and  $(q_1, p_1) = (1, 0)$ ,  $(q_k, p_k) = (q, p)$ .*

*Proof.* The proof is the successive application of Lemma 4.2.  $\square$

**Corollary 4.4.** *Any lens space  $L(p; q)$  is  $T^2$ -equivariantly oriented boundary.*

*Proof.* Without loss of generality, we may assume that  $0 < q < p$ . So by Lemma 4.3, there exists  $(q_1, p_1), \dots, (q_k, p_k) \in \mathbb{Z}^2$  where  $|\det\{(q_i, p_i), (q_{i+1}, p_{i+1})\}| = 1$  for all  $i = 1, \dots, k-1$  and  $(q_1, p_1) = (1, 0)$ ,  $(q_k, p_k) = (-q, p)$ . Consider the  $(k+1)$ -gon  $Q^2$  with vertices  $V_1, \dots, V_{k+1}$ . So the edges are  $V_1V_2, \dots, V_kV_{k+1}, V_{k+1}V_1$ . Define a function  $\eta : \{V_iV_{i+1} \mid i = 1, \dots, k\} \rightarrow \mathbb{Z}^2$  by  $\eta(V_iV_{i+1}) = (q_i, p_i)$  for  $i = 1, \dots, k$ . Let  $T_F$  be the circle subgroup of  $T^2$  determined by  $\eta(V_iV_{i+1})$  if  $F = V_iV_{i+1}$  and  $T_F = T^2$  if  $F = \{V_i\}$  for  $i = 2, \dots, k$ . We define an equivalence relation  $\sim_b$  on the product  $T^2 \times Q$  by

$$(4.1) \quad (t, x) \sim_b (s, y) \text{ if } x = y \text{ and } ts^{-1} \in T_F$$

where  $F \neq V_1V_{k+1}$  is the unique face containing  $x$  in its relative interior. We denote the quotient space  $(T^2 \times Q^2) / \sim_b$  by  $W(Q^2, \eta)$ . The space  $W(Q^2, \eta)$  is an oriented manifold with boundary and the boundary is  $(T^2 \times V_1V_{k+1}) / \sim_b$  which is equivariantly diffeomorphic to  $L(p; q)$ .  $\square$

**4.2. Cobordism of  $L(p; q_1, q_2)$ , that is when  $n = 2$ .** Consider a 2-simplex  $\Delta^2$  with vertices  $v_0, v_1, v_2$  and edges  $f_0 = v_1v_2, f_1 = v_0v_2$  and  $f_2 = v_0v_1$ . Let  $p, q_1$  and  $q_2$  be integers with  $0 < q_1 \leq q_2 < p$  such that  $\gcd\{p, q_i\} = 1$  for  $i = 1, 2$ . Then the function  $\xi : \mathcal{F}(\Delta^2) \rightarrow \mathbb{Z}^3$  defined by  $\xi(f_0) = (-q_1, -q_2, p)$ ,  $\xi(f_1) = (1, 0, 0)$  and  $\xi(f_2) = (0, 1, 0)$  is a hyper characteristic function of  $\Delta^2$  (see Figure 1 (b)) and the manifold  $L(\Delta^2, \xi)$  is the lens space  $L(p; q_1, q_2)$ , see Section 2.

Consider a 3-simplex  $\Delta^3$  with vertices  $\{V_0, V_1, V_2, V_3\}$  and facets  $\{F_i = V_0 \dots \widehat{V_i} \dots V_3\}$  opposite to the vertex  $V_i$  for  $i = 0, 1, 2, 3$ . We want to extend the hyper characteristic function  $\xi$  of  $\Delta^2$  to a rational characteristic function  $\eta : \mathcal{F}(\Delta^3) \rightarrow \mathbb{Z}^3$  of  $\Delta^3 = V_0V_1V_2V_3$  such that  $\eta(F_0) = \xi(f_0) = (-q_1, -q_2, p)$ ,  $\eta(F_1) = \xi(f_1) = (1, 0, 0)$  and  $\eta(F_2) = \xi(f_2) = (0, 1, 0)$ . Namely, we have to define  $\eta(F_3) = (a, b, c) \in \mathbb{Z}^3$  so that  $\{\eta(F_i), \eta(F_j)\}$  forms a part of basis of  $\mathbb{Z}^3$  for any  $i, j = 0, 1, 2, 3$  with  $i \neq j$ . It is elementary to check that such defined  $\eta$  is a rational characteristic function of  $\Delta^3$  if and only if

$$(4.2) \quad \gcd\{a, c\} = 1, \gcd\{b, c\} = 1 \text{ and } \gcd\{bp + q_2c, -(ap + q_1c), bq_1 - aq_2\} = 1.$$

In the following theorem, we would like to choose  $a, b \in \{0, -1\}$  and  $c = 1$  in order to make the following discussion simple. Namely, if  $(a, b, c) = (0, 0, 1)$ , then the equation 4.2 is equivalent to  $\gcd\{q_1, q_2\} = 1$ . If  $(a, b, c) = (0, -1, 1)$ , then the equation 4.2 is equivalent to  $\gcd\{q_1, p - q_2\} = 1$ . If  $(a, b, c) = (-1, 0, 1)$ , then the equation 4.2 is equivalent to

$\gcd\{p - q_1, q_2\} = 1$ . If  $(a, b, c) = (-1, -1, 1)$ , then the equation 4.2 is equivalent to  $\gcd\{p - q_1, p - q_2\} = 1$ . From the above observation, we raise the following question.

**Question 4.5.** Let  $\mathcal{L}(3) = \{(q_1, q_2, p) \in \mathbb{Z}^3 : \gcd\{p, q_1\} = \gcd\{p, q_2\} = 1 \text{ and } 0 < q_1 \leq q_2 < p\}$ . Suppose  $(q_1, q_2, p) \in \mathcal{L}(3)$ . Do one of the following condition hold?

- (1)  $\gcd\{q_1, q_2\} = 1$ ,
- (2)  $\gcd\{p - q_1, q_2\} = 1$ ,
- (3)  $\gcd\{q_1, p - q_2\} = 1$  and
- (4)  $\gcd\{p - q_1, p - q_2\} = 1$ .

**Remark 4.6.** The Question 4.5 is true if  $q_1$  is relatively prime to  $q_2$ . Let  $(q_1, q_2, p) \in \mathcal{L}(3)$  with  $q_2 = q_1 k_1$  for some  $k_1 \in \mathbb{Z}$ . Since  $p$  is relatively prime to  $q_1$ ,  $p - q_2$  is relatively prime to  $q_1$ . Hence condition (3) holds in Question 4.5. Note that if for  $(q_1, q_2, p) \in \mathcal{L}(3)$  one of the conditions in Question 4.5 holds then the corresponding hyper characteristic function of  $\Delta^2$  can be extend to a rational characteristic function of  $\Delta^3$ .

Let  $A(n) = \{(q_1, q_2, p) \in \mathcal{L}(3) : n = q_1 + q_2 + p\}$ . It is checked that Question 4.5 is true if  $(q_1, q_2, p) \in A(n)$  for all  $0 < n \leq 50$ . Let  $\mathfrak{N} := \max\{n \in \mathbb{Z} : \text{for any } (q_1, q_2, p) \in A(k) \text{ Question 4.5 is true for all } k \leq n\}$ .

**Theorem 4.7.** The lens space  $L(p; q_1, q_2)$  is  $T^3$ -equivariantly the boundary of an oriented manifold if  $(q_1, q_2, p) \in A(n)$  for  $n \leq \mathfrak{N}$ .

*Proof.* Let  $\Delta^3$  be a 3-simplex with vertices  $\{V_0, V_1, V_2, V_3\}$ . Let  $F_i$  be the facet of  $\Delta^3$  opposite to the vertex  $V_i$  for  $i = 0, 1, 2, 3$ . From the definition of  $A(n)$ , we can choose  $(a, b, c) = (\epsilon_1, \epsilon_2, 1)$  with  $\epsilon_1, \epsilon_2 \in \{0, -1\}$  such that the function  $\eta : \mathcal{F}(\Delta^3) \rightarrow \mathbb{Z}^3$  defined by  $\eta(F_0) = (-q_1, -q_2, p)$ ,  $\eta(F_1) = (1, 0, 0)$ ,  $\eta(F_2) = (0, 1, 0)$  and  $\eta(F_3) = (\epsilon_1, \epsilon_2, 1)$  is a rational characteristic function of  $\Delta^3$ . By Lemma 3.3 we get an oriented  $T^3$ -manifold  $W(\Delta_V^3, \eta)$  with boundary. The boundaries of  $W(\Delta_V^3, \eta)$  are the lens spaces  $(T^3 \times \Delta_{V_i}^2)/\sim_b$  for  $i = 0, 1, 2, 3$  where  $\sim_b$  is the equivalence relation in (3.1). The simple polytope  $\Delta_V^3$  and the rational characteristic function  $\eta$  is given in the Figure 4.

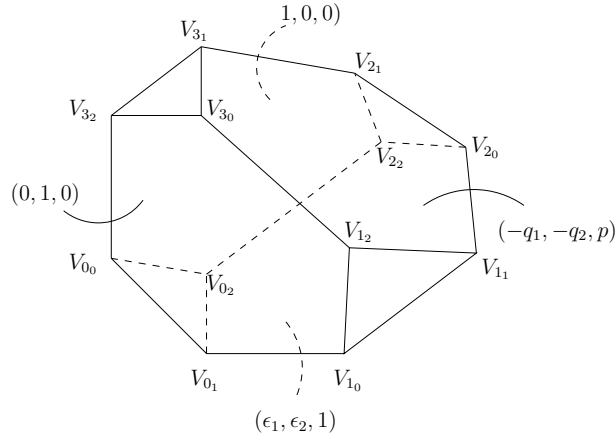


FIGURE 4. Vertex cut  $\Delta_V^3$  of  $\Delta^3$ .

Let  $\Delta_{V_i}^2 = V_{i0}V_{i1}V_{i2}$  be the facet of  $\Delta_V^3$  obtained from the vertex cut of  $\Delta^3$  at the vertex  $V_i$ , where  $V_{ij} = \Delta_{V_i}^2 \cap F_k \cap F_l$  with  $\{i, j, k, l\} = \{0, 1, 2, 3\}$ . Let  $\xi^i : \mathcal{F}(\Delta_{V_i}^2) \rightarrow \mathbb{Z}^3$  be the hyper characteristic function defined by  $\xi^i(V_{ij}V_{ik}) = \eta(F_l)$  where  $\{i, j, k, l\} = \{0, 1, 2, 3\}$ .

When  $i = 0$ , because  $\text{Im}(\xi^0) = \{\eta(F_1), \eta(F_2), \eta(F_3)\}$  is a basis of  $\mathbb{Z}^3$ , the boundary component  $L(\Delta_{V_0}^2, \xi^0) = (T^3 \times \Delta_{V_0}^2)/\sim_b$  of  $W(\Delta_V^3, \eta)$  is weakly-equivariantly diffeomorphic to the sphere  $S^5$ , which is a  $T^3$ -equivariant boundary.

When  $i = 1$ ,  $\text{Im}(\xi^1) = \{\eta(F_0), \eta(F_2), \eta(F_3)\}$  and clearly  $|\det[\eta(F_0), \eta(F_2), \eta(F_3)]| = |\epsilon_1 p + q_1| < p$ . Consider the automorphism of  $T^3$  determined by

$$(1, 0, 0) \rightarrow (0, 0, 1), (0, 1, 0) \rightarrow (0, 1, 0) \text{ and } (\epsilon_1, \epsilon_2, 1) \rightarrow (1, 0, 0).$$

So,  $(-q_1, -q_2, p) \rightarrow (p, -q_2 - \epsilon_2 p, -q_1 - \epsilon_1 p)$ . Let  $p \equiv q_{1_1} \pmod{|\epsilon_1 p + q_1|}$  where  $q_{1_1} = q_1$  if  $\epsilon_1 = -1$  and  $0 < q_{1_1} < q_1$  if  $\epsilon_1 = 0$ . Let  $\epsilon_2 p + q_2 \equiv q_{1_2} \pmod{|\epsilon_1 p + q_1|}$  where  $q_{1_2} = q_2$  if  $(\epsilon_1, \epsilon_2) = (-1, 0)$ ,  $q_{1_2} = q_2 - q_1$  if  $(\epsilon_1, \epsilon_2) = (-1, -1)$  and  $0 < q_{1_2} < q_1$  if  $\epsilon_1 = 0$ . Hence the boundary component  $L(\Delta_{V_1}^2, \xi^1) = (T^3 \times \Delta_{V_1}^2)/\sim_b$  of  $W(\Delta_V^3, \eta)$  is  $\delta_1$ -equivariantly diffeomorphic to the lens space  $L(|\epsilon_1 p + q_1|; q_{1_1}, q_{1_2})$  for some automorphism  $\delta_1$  of  $T^3$ . Clearly,  $0 < |q_{1_1}| \leq |q_{1_2}| < |\epsilon_1 p + q_1|$ ,  $\gcd\{\epsilon_1 p + q_1, q_{1_1}\} = 1 = \gcd\{\epsilon_1 p + q_1, q_{1_2}\}$  and  $(|q_{1_1}|, |q_{1_2}|, |\epsilon_1 p + q_1|) \in A(k_1)$  where  $k_1 = |q_{1_1}| + |q_{1_2}| + |\epsilon_1 p + q_1| < q_1 + q_2 + p < \mathfrak{N}$ .

When  $i = 2$ ,  $\text{Im}(\xi^2) = \{\eta(F_0), \eta(F_1), \eta(F_3)\}$  and  $|\det[\eta(F_0), \eta(F_1), \eta(F_3)]| = |\epsilon_2 p + q_2| < p$ . Similarly to the previous case we can show that the boundary component  $L(\Delta_{V_2}^2, \xi^2) = (T^3 \times \Delta_{V_2}^2)/\sim_b$  of  $W(\Delta_V^3, \eta)$  is  $\delta_2$ -equivariantly diffeomorphic to the lens space  $L(|\epsilon_2 p + q_2|; q_{2_1}, q_{2_2})$  for some integers  $q_{2_1}, q_{2_2}$  with  $0 < |q_{2_1}| \leq |q_{2_2}| < |\epsilon_2 p + q_2|$ . Here  $\epsilon_2 p + q_2$  is relatively prime to both  $q_{2_1}$  and  $q_{2_2}$ , and  $(|q_{2_1}|, |q_{2_2}|, |\epsilon_2 p + q_2|) \in M(k_2)$  with  $k_2 = |q_{2_1}| + |q_{2_2}| + |\epsilon_2 p + q_2| < \mathfrak{N}$ .

When  $i = 3$ ,  $\text{Im}(\xi^3) = \{\eta(F_0), \eta(F_1), \eta(F_2)\}$  and  $|\det[\eta(F_0), \eta(F_1), \eta(F_2)]| = p$ . Hence the boundary component  $L(\Delta_{V_3}^2, \xi^3) = (T^3 \times \Delta_{V_3}^2)/\sim_b$  of  $W(\Delta_V^3, \eta)$  is the lens space  $L(p, q_1, q_2)$  with the natural action  $\delta_3$  of  $T^3$ .

From the above discussion we get that the  $T^3$ -equivariant cobordism class  $[L(p; q_1, q_2)]_{\delta_3}$  of  $L(p; q_1, q_2)$  is equal to  $[L(|\epsilon_2 p + q_2|; q_{2_1}, q_{2_2})]_{\delta_2} + [L(|\epsilon_1 p + q_1|; q_{1_1}, q_{1_2})]_{\delta_1}$  where  $\delta_i$ 's are the corresponding actions of  $T^3$  on the respective lens spaces and  $0 < |\epsilon_1 p + q_1|, |\epsilon_2 p + q_2| < p$ . So continuing the previous technique, we can show that  $[L(p; q_1, q_2)]_{\delta_3}$  is zero. Actually, we can construct an oriented  $T^3$ -manifold with boundary where the boundary is the lens space  $L(p; q_1, q_2)$  by gluing the successive corresponding boundaries via equivariant maps.  $\square$

**Corollary 4.8.** *The lens space  $L(p; q_1, q_2)$  is  $T^3$ -equivariantly the boundary of an oriented manifold if two integers  $q_1$  and  $q_2$  are relatively prime.*

*Proof.* Since  $q_1$  and  $q_2$  are relatively prime, the condition (1) in Question 4.5 holds. We may assume  $0 < q_1 \leq q_2 < p$ . (Here the equality  $q_1 = q_2$  holds when they are equal to 1.) In this case we consider  $(\epsilon_1, \epsilon_2, 1) = (0, 0, 1)$ . Let  $p \equiv q_{1_1} \pmod{q_1}$  with  $0 < q_{1_1} < q_1$ ,  $q_2 \equiv q_{1_2} \pmod{q_1}$  with  $0 < q_{1_2} < q_1$ . Then the boundary component  $(T^3 \times \Delta_{V_1}^2)/\sim_b$  of  $W(\Delta_V^3, \xi)$  is  $\delta_1$ -equivariantly diffeomorphic to the lens space  $L(q_1; p, q_2) = L(q_1; q_{1_1}, q_{1_2})$  for some automorphism  $\delta_1$  of  $T^{n+1}$ . Similarly let  $p \equiv q_{2_1} \pmod{q_2}$  with  $0 < q_{2_1} < q_2$ . Then the boundary component  $(T^3 \times \Delta_{V_2}^2)/\sim_b$  is  $\delta_2$ -equivariantly diffeomorphic to the lens space  $L(q_2; q_{2_1}, q_1)$  and the boundary component  $(T^3 \times \Delta_{V_3}^2)/\sim_b$  is  $\delta_3$ -equivariantly diffeomorphic to the lens space  $L(p; q_1, q_2)$  for some automorphisms  $\delta_2$  and  $\delta_3$  of  $T^{n+1}$ . Note that  $\gcd\{q_1, q_{1_1}\} = 1 = \gcd\{q_1, q_{1_2}\}$  and  $\gcd\{q_2, q_{2_1}\} = 1 = \gcd\{q_2, q_1\}$  with  $0 < q_{1_1}, q_{1_2} < p$  and  $0 < q_{2_1} < q_2 < p$ . So  $[L(p; q_1, q_2)]_{\delta_3} = [L(q_1; q_{1_1}, q_{1_2})]_{\delta_1} + [L(q_2; q_{2_1}, q_1)]_{\delta_2}$ . Then applying the similar process inductively to get the corollary.  $\square$

**Corollary 4.9.** *The lens space  $L(p; q_1, q_2)$  is  $T^3$ -equivariantly the boundary of an oriented manifold if  $q_2 = kq_1$ ,  $q_1 + q_2 + p > \mathfrak{N}$  and  $p + q_1 < \mathfrak{N}$ .*

*Proof.* Since  $(q_1, q_2, p) = (q_1, q_1 k, p) \in \mathcal{L}(3)$ , the condition (2) in Question 4.5 holds. So we consider  $(\epsilon_1, \epsilon_2, 1) = (0, -1, 1)$  in Theorem 4.7. We may assume  $0 < q_1 \leq q_2 < p$ . Let  $p \equiv q_{1_1} \pmod{p}$  with  $0 < q_{1_1} < q_1$ . Hence the boundary components  $(T^3 \times \Delta_{V_1}^2)/\sim_b$ ,  $(T^3 \times \Delta_{V_2}^2)/\sim_b$  and  $(T^3 \times \Delta_{V_3}^2)/\sim_b$  of  $W(\Delta_V^3, \xi)$  are  $\delta_1$ -,  $\delta_2$ - and  $\delta_3$ -equivariantly diffeomorphic to the lens spaces  $L(q_1; p, p - q_2) = L(q_1; q_{1_1}, p - q_2)$ ,  $L(p - q_2; p, q_1) = L(p - q_2; q_2, q_1)$  and  $L(p; q_1, q_2)$  for some automorphisms  $\delta_1$ ,  $\delta_2$  and  $\delta_3$  of  $T^3$  respectively. So  $[L(p; q_1, q_2)]_{\delta_3} =$

$[L(q_1; q_{1_1}, p - q_2)]_{\delta_1} + [L(p - q_2; q_2, q_1)]_{\delta_2}$ . Observe that  $q_1 + q_{1_1} + p - q_2 < q_1 + q_1 + p - q_1 = p + q_1 < \mathfrak{M}$  and  $q_1 + q_2 + p - q_2 = p + q_1 < \mathfrak{M}$ . Then applying the Theorem 4.7, we get the corollary.  $\square$

**4.3. Cobordism of  $L(p; q_1, \dots, q_n)$  when  $n > 2$ .** Let  $\Delta^n$  be an  $n$ -simplex with vertices  $\{V_0, V_1, \dots, V_n\}$ , and let  $F_i$  be the facet which does not contain the vertex  $A_i$  for  $i = 0, \dots, n$ . Let  $\{e_i \mid i = 1, \dots, n+1\}$  be the standard basis of  $\mathbb{Z}^{n+1}$ . Define a function  $\xi : \{F_i \mid i = 0, \dots, n\} \rightarrow \mathbb{Z}^{n+1}$  by  $\xi(F_0) = (-q_1, \dots, -q_n, p)$  and  $\xi(F_i) = e_i$  for  $i = 1, \dots, n$  where  $p$  is relatively prime to each  $q_i$  for  $i = 1, \dots, n$  with  $0 \leq q_1 \leq \dots \leq q_n < p$ . From Section 2 we get  $L(\Delta^n, \xi) = L(p; q_1, \dots, q_n)$ . Let  $\mathcal{L}(n) = \{(q_1, \dots, q_n, p) \in \mathbb{Z}^{n+1} : p \text{ is relatively prime to } q_1, \dots, q_{n-1} \text{ and } q_n \text{ with } 0 \leq q_1 \leq \dots \leq q_n < p\}$ .

**Question 4.10.** Suppose  $(q_1, q_2, \dots, q_n, p) \in \mathcal{L}(n)$ . Do there exist  $(\epsilon_1, \dots, \epsilon_n) \in \{0, -1\}^n$  such that each of the following equations has an integral solution in  $\mathbb{Z}^3$ ,

- (1)  $x_1(\epsilon_2 p + q_2) - y_1(\epsilon_1 p + q_1) + z_1(\epsilon_1 q_2 - \epsilon_2 q_1) = 1,$
- (2)  $x_2(\epsilon_3 p + q_3) - y_2(\epsilon_1 p + q_1) + z_2(\epsilon_1 q_3 - \epsilon_3 q_1) = 1,$
- (3)  $x_3(\epsilon_4 p + q_4) - y_3(\epsilon_1 p + q_1) + z_3(\epsilon_1 q_4 - \epsilon_4 q_1) = 1,$
- (4) ...
- (5)  $x_k(\epsilon_n p + q_n) - y_k(\epsilon_{n-1} p + q_{n-1}) + z_k(\epsilon_{n-1} q_n - \epsilon_n q_{n-1}) = 1.$

where  $k = n(n+1)/2$ .

**Remark 4.11.** Question 4.10 is true if  $q_i$  is relatively prime to  $q_j$  for  $i \neq j$  and  $q_l = q_1 k_l$  for some  $k_l \in \mathbb{Z}$  for all  $l = 1, \dots, n$ .

Let  $\mathfrak{M}(m) = \{(q_1, q_2, \dots, q_n, p) \in \mathcal{L}(n) \mid m = q_1 + q_2 + \dots + q_n + p\}$ . Let  $\mathfrak{M} = \max\{m \in \mathbb{Z} \mid \text{for any } (q_1, \dots, q_2, p) \in A(k) \text{ Question 4.10 is true for all } k \leq m\}$ .

**Theorem 4.12.** The lens space  $L(p; q_1, \dots, q_n)$  is  $T^{n+1}$ -equivariantly the boundary of an oriented manifold if  $(q_1, \dots, q_n, p) \in \mathfrak{M}(m)$  with  $m \leq \mathfrak{M}$ .

*Proof.* Let  $\Delta^{n+1}$  be the  $(n+1)$ -dimensional simplex with vertices  $V_0, V_1, \dots, V_{n+1}$  and facets  $F_0, \dots, F_{n+1}$  where  $F_i$  does not contain the vertex  $V_i$  for  $i = 0, \dots, n+1$ . Let  $\{e_1, \dots, e_{n+1}\}$  be the standard basis of  $\mathbb{Z}^{n+1}$ . Let  $(\epsilon_1, \dots, \epsilon_n) \in \mathbb{Z}_2^n$  such that equations in Question 4.10 have integral solutions. We define a function  $\eta : \{F_0, \dots, F_{n+1}\} \rightarrow \mathbb{Z}^{n+1}$  by

$$\eta(F_0) = (-q_1, \dots, -q_n, p), \eta(F_{n+1}) = (\epsilon_1, \dots, \epsilon_n, 1) \text{ and } \eta(F_i) = e_i \text{ for } i = 1, \dots, n.$$

Since  $p$  is relatively prime to each  $q_i$  for  $i = 1, \dots, n$ ,  $\eta$  is an rational characteristic function of  $\Delta^{n+1}$ . Hence by Lemma 3.3,  $W(\Delta_{V_0}^{n+1}, \eta)$  is an oriented  $T^{n+1}$ -manifold with boundary where the boundaries are the generalized lens spaces  $(T^{n+1} \times \Delta_{V_i}^n) / \sim_b$  for  $i = 0, 1, \dots, n+1$ . Clearly the facets of  $\Delta_{V_i}^n$  are  $\{\Delta_{V_i}^n \cap F_j : j \neq i \text{ and } j \in 0, \dots, n+1\}$ . The restriction of  $\eta$  on the facets of  $\Delta_{V_i}^n$  is given by

$$\xi^i(\Delta_{V_i}^n \cap F_j) = \eta(F_j) \text{ } j \neq i \text{ and } j \in 0, \dots, n+1.$$

So  $\xi^i$  is a hyper characteristic function of  $\Delta_{V_i}^n$  and  $(T^{n+1} \times \Delta_{V_i}^n) / \sim_b = L(\Delta_{V_i}^n, \xi^i)$ . Similarly as in the proof of Theorem 4.7, we can show that  $L(\Delta_{V_{n+1}}^n, \xi^{n+1}) \cong L(p; q_1, \dots, q_n)$ ,  $L(\Delta_{V_0}^n, \xi^0) \cong S^{2n+1}$  and  $L(\Delta_{V_i}^n, \xi^i)$  is  $\delta_i$ -equivariantly diffeomorphic to the lens spaces  $L(|\epsilon_i p - q_i|; q_1^i, \dots, q_n^i)$  for  $i = 1, \dots, n$  where  $0 \leq |q_1^i| \leq \dots \leq |q_n^i| < |\epsilon_i p - q_i| < p$  with  $|q_1^i| + \dots + |q_n^i| + |\epsilon_i p - q_i| < \mathfrak{M}$ . Continuing this process, we can show  $L(p; q_1, \dots, q_n)$  is  $T^{n+1}$ -equivariantly boundary of an oriented manifold.  $\square$

**Corollary 4.13.** The lens space  $L(p; q_1, \dots, q_n)$  is equivariantly the boundary of an oriented manifold if any two integers of  $\{q_1, \dots, q_n\}$  are relatively prime.

*Proof.* Since any two integers of  $\{q_1, \dots, q_n, p\}$  are relatively prime, the equations in Question 4.10 have integral solutions when  $(\epsilon_1, \dots, \epsilon_n) = (0, \dots, 0)$ . We may assume that  $0 < q_1 \leq \dots \leq q_n < p$  for all  $i = 1, \dots, n$ . We consider  $\eta(F_{n+1}) = (\epsilon_1, \dots, \epsilon_n, 1) = (0, \dots, 0, 1)$ . Hence we can show that  $L(\Delta_{V_{n+1}}^n, \xi^{n+1}) \cong L(p; q_1, \dots, q_n)$ ,  $L(\Delta_{V_0}^n, \xi^0) \cong S^{2n+1}$  and  $L(\Delta_{V_i}^n, \xi^i)$  is  $\delta_i$ -equivariantly diffeomorphic to  $L(q_i; q_1^i, \dots, q_n^i)$  for  $i = 1, \dots, n$  where  $0 < q_1^i = q_j - k_j q_i < q_i$  if  $j \neq i$  and  $0 < q_i^i = p - k_i q_1 < q_i$  for some  $0 \leq k_j \in \mathbb{Z}$ . Observe that any two integers of  $\{q_1^i, \dots, q_n^i, q_i\}$  are relatively prime with  $q_i < p$  for  $i = 1, \dots, n$ . Continuing this process, we can show  $L(p; q_1, \dots, q_n)$  is  $T^{n+1}$ -equivariantly boundary of an oriented manifold.  $\square$

**Remark 4.14.** The oriented cobordism class of  $L(p; q_1, \dots, q_n)$  is zero, since all the Stiefel-Whitney numbers of it are zero.

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